The Hyperbolic Geometry of SAR Imaging

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Abstract
This paper shows how we can use hyperbolic functions to write an exact mathematical representation of SAR imaging. This problem is primarily a geometric one, that of accounting for curved wavefronts: the spirit of this paper is to emphasize these geometrical properties over electromagnetic ones. This gives us a new and fruitful way to think about SAR imaging. Within this framework, I show how to correct for deviations of a radar from a straight flight path. This method will work even in situations where the curvature of the wavefronts is very large, where traditional methods do not. The image-formation algorithm, called $\omega-k$ migration, that results from this analysis of SAR imaging is simpler and faster than polar formatting, especially for radars with very large beamwidths—as they will at very low frequencies. As an added benefit, $\omega-k$ migration is surprisingly simple to derive.

1. Introduction

Historically, processing synthetic aperture radar (SAR) data to produce images has used an approximation to express the instantaneous distance from the radar to a point on the ground. While this approximation is often adequate, it is still an approximation: we would prefer to have an exact expression both to aid our understanding of the SAR imaging problem and to improve the accuracy of our computations. In this paper, I derive an exact mathematical representation of the SAR-imaging problem. We should be pleased to see that the result is very simple. In particular, I shall show how, within this framework, we can make exact corrections for lateral motions of the radar.

The usual SAR-imaging approximation ignores the curvature of the wavefronts: this leads directly to the now-familiar polar formatting (see Walker, 1980; Jakowitz et al., 1996, provide a good discussion of SAR processing using polar formatting). When the SAR antenna has a narrow beamwidth, this approximation may be adequate, but it breaks down when the beamwidth is large, as it will be for low-frequency SAR’s of the type that are being studied for foliage-penetrating applications. However, using hyperbolic geometry, we can develop expressions that take the curvature into account exactly: no approximation is necessary. This leads to an alternative method of producing SAR images, which is called $\omega-k$ migration (see Milman, 1993, and references cited therein; Prati et al., 1990; and Cafforio et al., 1989, 1991a and 1991b, are the original papers on the subject, as far as I know). Not only is this method more accurate than polar formatting, it is also simpler and easier to implement on a computer. There are many advantages, and no known drawbacks, to using $\omega-k$ migration.
Polar formatting has the limitation that, because of the plane-wave approximation, the focus of an image deteriorates with increasing distance from the scene center. This requires that we divide the image into many sub-images; motion-compensate each pulse to the center of each sub-image; form these separate sub-images; and, finally, patch them together (Carrara et al., 1995b). I show this in Figure 1a. If there are \( N \) sub-images, the data need to be processed \( N \) times. Aside from the extra computation costs, this introduces discontinuities in the phase of the complex image; these discontinuities will affect algorithms that use this complex phase. We shall see that \( \omega-k \) migration processing has none of these drawbacks.

One way to view migration processing is to consider, as I do here, the geometrical aspects of the problem, rather than concentrating on the electromagnetic formalism. Central to this issue is the question of how rotating the scene being imaged affects the signal history—since the scene appears to rotate as the SAR flies by it—and how rotation affects its Fourier transform. Consider a function \( f(x,y) \) and its Fourier transform \( F(r,s) \). A rotation of \( f \) in the \( x-y \) plane produces the same rotation of \( F \) in the \( r-s \) plane (I discuss this further in section 5). If we can neglect the curvature of the wavefronts, we can use this property of Fourier transforms to develop a polar-formatting algorithm (see Walker, 1980).

In polar formatting, the pulses are laid out in a radial manner, with range frequency represented in the radial direction, going from \( f_0 \) to \( f_0 + B \), the minimum and maximum frequencies in the chirp (\( i.e., B \) is the bandwidth of the radar; see Figure 2). The direction changes to reflect the changing direction from the scene being imaged to the radar, so the angular coördinate \( \phi \) in this diagram corresponds to the direction from the SAR to the scene center. The curvature of the lines of constant range frequency, in this picture, depends on the ratio of the bandwidth \( B \) to the center frequency \( f_0 + \frac{1}{2}B \), and the total angle subtended by the SAR flight path. However, because the wavefronts are curved, the resulting image is only properly focused at the center; the defocusing near the edges limits the size of the image that can be formed.

In this discussion of migration processing, I shall start with the data as representing the reflected power as a function of distance from the SAR to the target (\( \rho \)) and of pulse
number, or the position of the SAR along the flight track \((t)\). In particular, I will neglect the fact that most radars are chirped—\(i.e.\) the frequency increases (or decreases) linearly during a pulse—and assume that the radar in question transmits pulses with a single wavelength \(\lambda\). In Milman (1993), I showed that the case of a chirped radar can be reduced to this simpler case. In practice, this is done by de-chirping the returned signal. (\(E.g.,\) see Franceschetti and Lanari, 1999, Chapter 1.)

Carrara et al. (1995a, pp. 401 ff) and Golden et al. (1994) discuss a similar method that they call \textit{range migration}. Although it is based, in part, on Milman (1993), they do not make use of the property that the migration formalism is an exact representation of the SAR imaging problem. This may be, in part, because they have not adopted the geometrical outlook I espouse here. Franceschetti and Lanari (1999, pp 125-126 and 274 ff) also employ the migration method, in that they make use of the Stolt transform described below, but they do not make use of the formalism that I present here. In addition, Franceschetti and Schirinzi (1990) and Franceschetti et al. (1996) both avoid the plane-wave approximation (which is equivalent to approximating the distance \(R\) to a target by a one-term Taylor series), and they both use a stationary-phase approximation in creating SAR images. However, their work does not provide the geometrical insight that can be obtained from the method I describe here.

Let me give an analogy. Suppose that we were unaware of Newton’s law of gravitation, but we knew that, given two bodies separated by the distance \(R_0 + r\), where \(r \ll R_0\), the gravitational force between them is proportional to

\[
\frac{1}{R_0^2} - \frac{2r}{R_0^3} + \frac{3r^2}{R_0^4} - \cdots
\]
It might be possible to calculate some approximate planetary positions from this, but without knowing the exact relationship, that the gravitational force is proportional to $1/(R_0 + r)^2$, it would not have been possible to land a spaceship on the moon. For a similar reason, knowing this exact solution to the SAR-imaging problem is important for understanding it.

There is always a need to correct for lateral motions of the radar, i.e., known deviations from a straight flight path. This problem is especially severe in the case of low-frequency radars used for foliage-penetration applications, where the beamwidths can be very large. In this paper, I show how to correct for these motions globally, in the sense that, assuming that we know the deviations from the nominal, straight flight path, we can perform a single correction to the data that compensates for that motion everywhere within the field-of-view of the radar. By contrast, the “range migration” algorithm discussed by Carrara et al. (1995a), Golden et al. (1994), and Carrara et al. (1995b) (and also personal communication) compensates for lateral motions by creating sub-images with the data compensated to the center of the scene (see Figure 1a). This only accounts for the lateral motions of the radar approximately and requires that the same raw data be processed many times, once for each sub-image that the signal history contributes to. With $\omega$-$k$ migration, we can process the data once to produce the full scene, as shown in Figure 1b.

2. An overview of $\omega$-$k$ migration

2.1 Hankel Transforms.

Before I start the derivation of $\omega$-$k$ migration, I should show you where we’re going. I want to show that there is a closed-form expression that relates the signal history from the SAR to the distribution of targets on the ground, and that this expression has an analytical inverse. In so doing, I make use of Hankel transforms, which are similar in many ways to the more familiar Fourier transform. I discuss Hankel transforms in more detail in Appendix A.

We can write a Fourier transform as

$$G(\omega) = \int_{-\infty}^{\infty} K_F(x) g(x) \, dx,$$

where $g(x)$ is an arbitrary function and the Fourier kernel $K_F(x) = \exp\{-2\pi i \omega x\}$. This transform has an inverse

$$g(x) = \int_{-\infty}^{\infty} K_F^*(\omega) G(\omega) \, d\omega,$$

where $K_F^*$ is the complex conjugate of $K_F$. Similarly, a Hankel transform is
\[ G(\omega) = \int_{-\infty}^{\infty} K_H(\omega x) g(x) \, dx , \]  

where the Hankel kernel is

\[ K_H = x H_0^{(2)}(\omega x) , \]  

and where \( H_0^{(2)}(x) = J_0(x) - iY_0(x) \) is a Hankel function of zero order of the second kind; \( J_0 \) and \( Y_0 \) are Bessel functions of the first and second kind. This transform also has an inverse:

\[ g(x) = \int_{-\infty}^{\infty} K_H^*(\omega x) G(\omega) \, d\omega . \]  

Hankel transforms are less familiar than Fourier transforms but have similar properties.

Now suppose I have a function \( A(x) \) whose Fourier transform is \( a(\omega) \). If I measure \( a(\omega) \), it is trivial to calculate \( A(x) \): we just use an inverse Fourier transform. Similarly, if \( B(x) \) is the Hankel transform of \( b(\omega) \), we can calculate \( B \) from \( b \) with an inverse Hankel transform. We shall see later that we can use the usual Fast Fourier transform (FFT) software to calculate a Hankel transform, so it can be calculated as efficiently as a Fourier transform.

Suppose that \( s(t,\rho) \) is the measured signal history from a SAR, where \( \rho \) is distance from the radar to a reflector and \( t \), the (slow) time that measures the position of the SAR along the flight track (here, I shall define the units so that the speed of the SAR, \( \nu = 1 \)). Let \( S(\omega,k) \) be the 2D Fourier transform of \( s(t,\rho) \); I shall show that \( S(\omega,k_y) \) is a 2D transform of the scene reflectivity \( \sigma^0(x,y) \), where \( k_y^2 = k^2 - \omega^2 \). The change of the independent variable from \( k \) to \( k_y \) is an important part of this theory. This transform is a Fourier transform in \( x \), the along-track direction, and a Hankel transform in \( y \), the cross-track direction. Let me write the combined Fourier-Hankel kernel as

\[ K_{F-H} = e^{-2\pi ik_y} H_0^{(2)}(2\pi k_y) k_y . \]  

Then I shall show that

\[ S(\omega,k_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_{F-H}(\omega x,k_y) \sigma(x,y) \, dx \, dy . \]  

But if this is so, we can form a SAR image easily: the image is just

\[ I(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_{F-H}^*(\omega x,k_y) S(\omega,k_y) \, dk_y \, d\omega . \]
This is an exact relationship: I have not had to make any plane-wave or similar approximation to arrive at equation (8). Furthermore, this relationship is valid regardless of the width of the size of the region illuminated by the antenna; there is no loss of focus at the edges. I am always surprised at how simple this solution is.

It is customary to analyze many kinds of SAR imaging problems by writing the signal history as a Taylor series and keep the first two or three terms. This is often an analytical nightmare, and it requires us to justify ignoring all terms higher than a certain order. But because $\omega-k$ migration provides an exact analytical relationship between signal history $S(\omega,k_y)$ and the scene $\sigma(x,y)$, we can analyze SAR imaging problems quite easily within this framework. I provide one such analysis in this paper, pertaining to the question of how to correct for lateral motions of the SAR, but first, I shall provide a rigorous derivation of the results that I have claimed in this section.

### 2.2 Geometry and signal history for a straight flight path

Consider a radar mounted on an airplane that is used to make synthetic aperture radar (SAR) images of the earth’s surface. Something that is well known, but apparently often overlooked, is that the shape of the curve that describes the instantaneous distance from the SAR to a fixed point on the ground is an hyperbola. We can make use of this property to simplify the analysis of the SAR imaging problem; Figure 3 shows the relevant geometry. Here, $h$ is the height of the radar above the (plane) surface of the earth. The radar moves with velocity $v$ along the $x$-axis. For simplicity, I shall use units where $v = 1$; therefore, $t$ is a distance (as well as slow time). Assume that the time $t = 0$ corresponds to $x = 0$, so $t$ measures the position of the SAR along the flight track. Furthermore, the flight path is assumed to extend from $-t_0$ to $+t_0$.

I note, in passing, that I am making no assumption as to where the scene being imaged is in relation to the SAR: there is no restriction that it be at broadside. In the present discussion I treat the so-called spotlight mode, where the radar beam moves continuously, relative to the radar, during the data collection so that the illumination remains fixed on the scene being imaged. Although it is customary to include an antenna weighting function $w(x,y)$ explicitly, I shall include it in the definition of the complex reflectivity $\sigma^0$ to keep the notation simple. The extension to the stripmap mode, where the direction of the beam is fixed relative to the radar, is straightforward.
Hyperbolic Geometry of SAR Imaging

The distance from the SAR, located at the position \((t, 0, h)\), to a point at \((x, y, 0)\) on the surface is

\[
R^2 = h^2 + y^2 + (x - t)^2.
\]  

(9)

We can reduce this to an equivalent 2-dimensional problem with the distance in the ground plane given by

\[
\rho^2 = R^2 - h^2 = y^2 + (x - t)^2.
\]  

(10)

Then the signal history is

\[
s_\theta(t, \rho) = e^{- \frac{4\pi\rho}{\lambda} \int_C \sigma^0(x, y) \, ds},
\]  

(11)

where \(\sigma^0(x,y)\) is the complex radar backscattering coefficient at location \((x,y)\); \(\lambda\), the electromagnetic wavelength; and the curve \(C\) is the path defined by \(\rho^2 = y^2 + (x - t)^2\). The complex exponential in this equation represents the relative phase of the reflected signal.

The path of a point on the ground, in the frame of reference fixed to the airplane, is

\[
y^2 = \rho^2 - (x - t)^2,
\]  

(12)

which, of course, is the equation for an hyperbola. This suggests that we use hyperbolic functions to describe the geometry of SAR imaging. I shall show how this allows us to use an exact analytic description of the SAR geometry. This will lead to an algorithm for producing SAR images that is mathematically exact; this is, in part, a simplified derivation of the so-called \(\omega-k\) migration algorithm described by Milman (1993) and others.

The hyperbolic functions \(\sinh \psi\) and \(\cosh \psi\) are related by

\[
\cosh^2 \psi - \sinh^2 \psi = 1.
\]  

(13)

If we let

\[
\sinh \psi = \frac{x - t}{y},
\]  

(14)

then

\[
\rho^2 = y^2 + (x - t)^2 = y^2(1 + \sinh^2 \psi) = y^2 \cosh^2 \psi.
\]  

(15)

Figure 4 shows the geometric meaning of the angle \(\psi\). It shows the path of an object on the ground located at a distance \(y\) from the SAR track, as seen from the aircraft. Now, since \(y = \rho \text{sech} \psi\), we can write \(x\) and \(y\) in the form
In order to develop an exact representation the phase history using hyperbolic functions, we will need to transform the variables of integration from \((x,y)\) to \((\rho,\psi)\). We will need to use the relations \(\tanh^2 \psi + \text{sech}^2 \psi = 1\),

\[
\frac{d}{d\psi} \tanh \psi = \text{sech}^2 \psi \quad \text{and} \quad \frac{d}{d\psi} \text{sech} \psi = \text{sech} \psi \tanh \psi.
\] (18)

The Jacobian of this transformation is \(\rho \text{sech} \psi\). The complex signal history is

\[
s(t,\rho) = \frac{1}{\rho} s_0(t,\rho) = \int e^{\frac{4\pi i \rho}{\lambda}} \sigma^0(\rho \tanh \psi + t, \rho \text{sech} \psi) \text{sech} \psi \, d\psi.
\] (19)

The factor \(1/\rho\) was introduced to cancel the factor \(\rho\) in the Jacobian.

Now take the 2-dimensional Fourier transform of \(s(t,\rho)\):

\[
S(\omega, k) = \int \int e^{-2\pi i (k\rho + \omega t)} s(t,\rho) \, d\rho \, dt
\]

\[
= \int \int e^{\frac{4\pi i \rho}{\lambda}} e^{-2\pi i (k\rho + \omega t)} \sigma^0(\rho \tanh \psi + t, \rho \text{sech} \psi) \text{sech} \psi \, d\rho \, d\psi.
\] (20)

Transform this by letting \(y = \rho \text{sech} \psi\), so
\[ S(\omega,k) = \int \int \int e^{-2\pi i \left(\frac{k + 2}{\lambda}y\right) \cos \psi + \omega t} \sigma^0 (y \sinh \psi + t, y) dy dt. \]  

(21)

Now substitute \( x = t + y \sinh \psi \), so

\[ S(\omega,k) = \int \int \int e^{-2\pi i \left(\frac{k + 2}{\lambda}y\right) \cos \psi - \omega t} \sigma^0 (x, y) dy dx dy. \]  

(22)

### 2.3 Stolt and Hankel transforms

Now make the following substitutions: let

\[ k_y^2 = \left( k + \frac{2}{\lambda} \right)^2 - \omega^2, \quad k = k_y \cosh \phi, \text{ and } \omega = k_y \sinh \phi. \]  

(23)

Note that the radial wavenumber ranges from \( 2/\lambda - \frac{1}{2} \Delta k \) to \( 2/\lambda + \frac{1}{2} \Delta k \), where \( \Delta k = 1/\delta r \), and \( \delta r \) is the spacing in distance. This substitution transforms \( k \), the wavenumber in the radial direction, to \( k_y \), the wavenumber in the \( y \)-direction: this effectively removes the curvature of the wavefronts. Using the relationship \( \cosh \psi \cos \phi + \sinh \psi \sin \phi = \cosh (\psi + \phi) \) in equation (22), we get the simple result that

\[ S(\omega,k_y) = \int \int \int e^{-2\pi i k_y y} \cosh \phi \sigma^0 (x, y) dx dy dy. \]  

(24)

The transformation of \( S(\omega,k) \) to \( S(\omega,k_y) \) is a one-dimensional interpolation, called the Stolt transform.

Compare equation (38) in Appendix A, which is the integral representation of a Hankel function, to the integral over \( \psi \) in equation (24); this shows that equation (24) is a Hankel transform of the complex reflectivity \( \sigma^0 \). I describe the relevant properties of Hankel functions and Hankel transforms in Appendix A.

Knowing this, we can perform the integration over \( t \) analytically, using equation (38):

\[ S(\omega,k_y) = \int \int H_0^{(2)} (2\pi k_y) e^{-2\pi i k y} \sigma^0 (x, y) dx dy \]

\[ = \int \int H_0^{(2)} (2\pi k_y) e^{-2\pi i k} \frac{\sigma^0 (x, y)}{y} y dx dy. \]  

(25)

The inverse of equation (25) is
Hyperbolic Geometry of SAR Imaging

\[
\frac{1}{y} \sigma^0(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{y} H_0^{(1)}(2\pi y k_y) e^{2\pi i \sigma \omega} S(\omega, k_y) y k_y \, dk_y, d\omega.
\]  

(26)

We should note the factor \(\exp\{-2\pi i y k_y \cosh(\psi - \phi)\}\) in equation (25): to the extent that we can take the infinite limits of integration seriously, the value of \(\phi\) is immaterial. In reality, there is only a finite angle \(\psi\) that we integrate over, so the value of \(\phi\) will make a small difference. However, this is a limitation on our ability to form images from a finite amount of data, and will arise no matter how we process the data: it is not a shortcoming of the \(\omega-k\) migration method.

This shows that, without making any plane-wave approximation, it is possible to write the radar cross-section distribution in terms of the signal history. We can invert it by using a Fourier transform in the along-track direction, and a Hankel transform in the cross-track direction. The Hankel transform is unfamiliar, and (to my knowledge) no one has Fast Hankel Transform (FHT) software the way we have FFT’s. Fortunately, we can use the approximation given by equation (39) in equation (26), which is accurate as long as \(2\pi y k_y > 3\) (see Appendix A). So the Hankel transform we need to perform is, aside from division by \((2\pi y k_y)\), equivalent to an FFT, and we can use the usual FFT software.

The form of the Stolt transform given in equation (23) is different from the usual definition—e.g., in Milman (1993). Here, I have used \(k + 2/\lambda\), instead of just \(k\), to account formally for the factor \(\exp\{-4\pi i \rho / \lambda\}\) in equations (11) et seq. We should also note that we can arrive at the approximations for \(H_0^{(1)}\) in equation (39) by employing the method of stationary phase (see Milman, 1993), and we do not need to mention Hankel functions at all if we do not want to. Carrara et al. (1995a) and Franceschetti and Lanari (1999) do just this. However, it is useful to see how we can represent the SAR imaging problem exactly using a Hankel transform, even if it does not directly affect the computational scheme for creating SAR images.

3. Shift to Scene Center

As I have formulated this problem, the origin of the coördinate system is the center of the flight track. However, if we do not move the origin to the center of the scene being imaged, we increase the bandwidth of the signal enormously. To this end, we can shift the coördinate system to the point \((x_0, y_0)\) as follows. In equation (24), replace \(x\) and \(y\) with \(x - x_0\) and \(y - y_0\):

\[
S(\omega, k_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2\pi i \sigma \omega (x - x_0)} e^{-2\pi i (y - y_0) k_y} \cosh(\psi - \phi) \sigma^0(x + x_0, y + y_0) \, dx \, dy \, d\psi
\]

\[
= e^{2\pi i \sigma \omega x_0} e^{2\pi i y_0 k_y} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2\pi i \sigma \omega x} e^{-2\pi i y k_y} \cosh(\psi - \phi) \sigma^0(x, y) \, dx \, dy \, d\psi.
\]  

(27)
Here, I have used the approximation to the Hankel function given in equation (37), and used \( \sigma_0(x,y) \) to denote reflectivity pattern shifted so that it is centered at \((x_0,y_0)\). Carrara et al. (1995a) call this a “matched filter”, but I think that it is something of a misnomer. It is not a matched filter in the Wiener sense, but only a shift of the coördinate system. This is a necessary step in digital data processing because it reduces the bandwidth needed in the along-track direction (efforts like those of Golden et al. to reduce the bandwidth are unnecessary).

Finally, the image is found from the two-dimensional Fourier transform of \( S \):

\[
I(x,y) = \int \int k_y^2 S(\omega,k_y) e^{-2\pi i(\omega x_0 + k_y y_0)} e^{2\pi i \omega x_0} e^{2\pi i k_y y_0} d\omega dk_y .
\]  

This comes about because we use equation (39) to approximate \( H_0(2\pi y k_y) \) in equation (26).

The steps forming an image using \( \omega-k \) migration are shown in Figure 5.

**Figure 5.** Steps in \( \omega-k \) SAR processing.

<table>
<thead>
<tr>
<th>Step</th>
<th>Signal History ( s(t,\rho) )</th>
<th>( \rightarrow )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>2-D FFT</td>
<td>( \rightarrow )</td>
</tr>
<tr>
<td>2.</td>
<td>Shift the Origin</td>
<td>( e^{-2\pi i(\omega x_0 + k_y y_0)} k_y^2 S(\omega,k) } )</td>
</tr>
<tr>
<td>3.</td>
<td>Stolt Transform</td>
<td>( \rightarrow )</td>
</tr>
<tr>
<td>4.</td>
<td>2-D Inverse FFT</td>
<td>( \rightarrow )</td>
</tr>
<tr>
<td>5.</td>
<td></td>
<td>Image ( I(x,y) )</td>
</tr>
</tbody>
</table>

4. **Limits of Integration Over \( \psi \)**

In the integral in equation (24), the limits of the integration over \( \psi \) are formally \( \pm \infty \). In reality, the range of \( \psi \) is limited by the geometry of the data collection. However, these limits are obviously finite. Suppose we are imaging a scene centered at location \((x_0,y_0)\), with the distance to the center being \( \rho_0 = (x_0^2 + y_0^2)^{1/2} \). If we use a stationary-phase analysis, it is clear that the integral

\[
\int_{\psi_0}^{\psi_1} e^{-2\pi i k_y \cosh(\psi - \phi)} d\psi
\]

is zero unless the stationary point of the integrand—where \( \psi = \phi \)—is included in the interval \((\psi_1,\psi_2)\). This corresponds to the geometric requirement that
where \( \pm t_0 \) are the limits of the flight track. This follows directly from the definitions in equation (23). Therefore, the signal history corresponding to a single point target only occupies a restricted region of \((\omega, k)\) space. It is the extent of \(\omega\) in this region that determines the sampling requirement for the SAR data collection. This is illustrated in Figure 6; note that the azimuth wavenumber \(\omega\) is centered at 0, but the range wavenumber \(k\) is centered at \(2/\lambda\). For a target at \(0^\circ\), this region is centered at \(\omega = 0\). Figure 7 shows a target at a squint angle of \(5^\circ\).

Suppose that the length of the SAR track extends from \(-t_0\) to \(+t_0\), and there are \(N\) pulses during the collection. Then the pulse spacing is \(\delta x = 2t_0/N\), and the wavenumber \(\omega\) is in the interval

\[
-\frac{1}{2\delta x} \leq \omega < \frac{1}{2\delta x}.
\]

(31)

Similarly, if the range resolution is \(\delta\), the range wavenumber \(k\) is in the interval

\[
\frac{2}{\lambda} - \frac{1}{2\delta r} \leq k < \frac{1}{2\delta r} + \frac{2}{\lambda}.
\]

(32)

Equation (22) shows us that the radial wavenumber is centered at \(2/\lambda\). In general, \(\omega\) is centered at

\[
\omega_0 = \frac{2x_o}{\lambda\rho_0},
\]

(33)
where $x_0$ is the azimuth of the scene center, and $\rho_0$ the distance to the center of the flight track. The range of $\omega$ will be shifted by an amount that depends on the squint angle, as shown in Figure 7. In order to process the data efficiently, it is necessary to limit the processing to the part of the $\omega$-$k$ plane where $S(\omega,k)$ can be different from zero. So we will want to limit the processing to the region where $2/\lambda - \frac{1}{2}\Delta k < k < 2/\lambda + \frac{1}{2}\Delta k$, and $2x_0/\lambda \rho_0 - \frac{1}{2}\Delta \omega < \omega < 2x_0/\lambda \rho_0 + \frac{1}{2}\Delta \omega$. One way to do this is to shift the data in $\omega$-$k$ space along each line of constant $k$ by an amount that depends on $k$. To this end, let $\beta \equiv \omega_0 / (k + 2/\lambda)$, and shift $S(\omega,k)$ by the amount $\beta(k + 2/\lambda)$. Then, instead of equation (20), we would have

$$S(\omega,k) = \int \int \int e^{-2\pi i (\omega - \beta k')^2} 1_{\sigma^0}(\rho \tanh \psi + t, \rho \sech \psi) \sech \psi \, d\rho \, dt \, d\psi,$$  \hspace{1cm}(34)$$

where $k' = k + 2/\lambda$. It is straightforward to derive the appropriate formula that corresponds to equation (26).

Note that this is different from rotating the signal history in the $t$-$\rho$ plane. If we did that, we would not be able to solve equation (25) for $\sigma^0$. We can solve it, however, if we shift the data in the $\omega$ direction, as I showed here.

5. Correction for lateral motions

We have already seen how to correct for deviations in the vertical direction from a straight flight path. But what happens if the SAR drifts laterally from this straight path? Suppose that, at time $t$, the SAR is displaced laterally by an amount $-\eta(t)$. This is equivalent to having the scene move in the $y$-direction by $\eta(t)$. If we simply multiply each pulse by a complex phase to correct for these motions by shifting it toward or away the SAR by
an amount $-\eta$, we are making an error because, while the SAR was displaced in the $y$-direction, the correction is being applied in the radial direction—see Figure 9. For wide-beamwidth SAR’s, this is a serious problem. We can perform the correction properly as follows.

The central notion of this theory of SAR image formation is that equation (27) is exact, and that it has an exact mathematical inverse. As far as I can tell, only this problem can be solved exactly. If the radar moves in any way other than a straight line, no exact solution is possible. Therefore, to account for lateral motions of the radar, we need to break the flight path into straight segments. In general, each one will be at some angle $\alpha$ to the nominal flight path and we need to correct each segment of the signal history for this tilt. To do this, we need to know how to treat a rotation and translation of the scene denoted by $\sigma^0(x,y)$.

In general, if $F(\omega,k)$ is the 2-dimensional Fourier transform of some function $f(x,y)$, and we rotate $f$ in the $(x,y)$ plane according to

\[
\begin{bmatrix}
x' \\
y'
\end{bmatrix} = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix}
\]

then the Fourier transform is rotated similarly:

\[
\begin{bmatrix}
\omega' \\
k'
\end{bmatrix} = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix} \begin{bmatrix}
\omega \\
k
\end{bmatrix}
\]

(36)

It is not possible to rotate the signal history $s(t,\rho)$ directly in the $(t,\rho)$ plane. Instead, we can account for the rotation $\sigma^0(x,y)$, in the $(x,y)$ plane by calculating $S(\omega,k_y)$ and then rotating it in the $(\omega,k_y)$ plane. We have already seen in equation (27) how a translation of the scene $\sigma^0$ changes $S(\omega,k_y)$. [The reader can verify that, if we rotate the signal history in the $(t,\rho)$ according to equation (35), it is no longer possible to solve equation (22)].

We can follow these steps to correct for these lateral motions:

1. Divide the true SAR flight track into linear segments. Suppose the nominal flight line is the segment $AZ$, as shown in Figure 10. In each segment ($AB$, $BC$, and so forth) the scene being imaged has a different direction and distance, as compared with the corresponding section of the path $AZ$. The longer the segments are, the better the corrections will be.

2. For each segment determine the transformation that corresponds to moving that segment to the nominal flight path.

3. Use an FFT and the Stolt transform to calculate $S(\omega,k_y)$.

4. Rotate and translate $S(\omega,k_y)$.

5. Reverse the processes in step 4. This properly corrects each piece of signal history for the deviation from the nominal path.

6. Combine these pieces to produce a single signal history and process it to produce an image, as outlined in the last section.
It should be clear that, forming $S(\omega, k_y)$ from one segment of the signal history, we will lose some of the information that we could have had if we were able to process the entire signal history at once. Our ability to account for the rotation of each segment of the flight path—and to remove its effect—is limited by the length of each segment, just as the azimuth resolution would be if we made an image from that segment alone. In my opinion, this loss of information is unavoidable: the image will be degraded to some extent if the radar does not fly in a straight line, no matter what we do. This degradation is inevitable; it is not due the $\omega$-$k$ migration algorithm and cannot be avoided by using some other algorithm. This is not to say that the method outlined above is the only way, or even necessarily the best way, to correct for lateral motions. However, given a particular set of circumstances, we can use the insight we gained from the $\omega$-$k$ migration paradigm to devise the most practical method for making these corrections.

6. Conclusion

I have demonstrated here that there is an exact mathematical solution to the problem of forming a SAR image from the signal history, at least in the case where the radar flies along a straight path. While this is useful in its own right for deriving a simple imaging algorithm, it is also important for understanding the SAR-imaging problem in general, as it gives us a framework for correcting for the problems that arise in real life.

For example, there is a straightforward method for compensating for lateral motions of the radar. Like other aspects of $\omega$-$k$ migration processing, it is based on using the known, exact mathematical relationship between the scene being imaged and the signal history. Although the hyperbolic functions involved may seem strange at first, viewing the SAR imaging process in terms of these functions simplifies both the basic image-formation algorithm and making corrections for lateral motions. In particular, the $\omega$-$k$ migration algorithm is simpler than polar formatting and does not have the limitations on image size that are imposed on polar formatting by the plane-wave approximation.
Appendix A. Hankel Functions

We shall use the zero-order Hankel functions, which are defined as \( H_0^{(1)}(z) = J_0(z) + iY_0(z) \) and \( H_0^{(2)}(z) = J_0(z) - iY_0(z) \), where \( J_0 \) and \( Y_0 \) are the zero-order Bessel functions of the first and second kind. We shall use the integral representations

\[
H_0^{(1)}(z) = \frac{2}{\pi i} \int_0^\infty e^{iz \cosh t} \, dt , \quad (37)
\]

\[
H_0^{(2)}(z) = \frac{2}{\pi i} \int_0^\infty e^{-iz \cosh t} \, dt . \quad (38)
\]

Later, we will use the following asymptotic approximations to \( H_0^{(1)} \) and \( H_0^{(2)} \).

\[
H_0^{(1)}(x) \approx \sqrt{\frac{2}{\pi x}} e^{i(x - \frac{1}{4} \pi)} ,
\]

\[
H_0^{(2)}(x) \approx \sqrt{\frac{2}{\pi x}} e^{-i(x - \frac{1}{4} \pi)} . \quad (39)
\]

These approximations for \( H_0^{(1)}(x) \) and \( H_0^{(2)}(x) \) are very accurate for \( x > 3 \). We shall see that this will always obtain for any plausible SAR. In the event that we need a more accurate asymptotic expansion, a more elaborate one is given below.

We can write the inverse Hankel transform as follows. If \( F(r) \) is a function such that

\[
\int_{-\infty}^{\infty} \sqrt{r} F(r) \, dr < \infty , \quad (40)
\]

then

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(r) H_0^{(2)}(ur) r \, dr \, H_0^{(1)}(us) u \, du = F(s) . \quad (41)
\]

(This relationship is also true when \( H_0^{(1)}(ur)H_0^{(2)}(us) \) is replaced with \( C_0(ur)C_0(us) \) and \( C_0 \) is either of the Bessel functions \( J_0 \) or \( Y_0 \).) This is just a restatement of equations (3) and (5).

A more accurate asymptotic expansion is the following:

\[
H_n^{(1)}(z) = \sqrt{\frac{2}{\pi z}} e^{-\left(\frac{1}{4} + i\pi\right)} \left[ \sum_{k=0}^{n} (-1)^k c_n, k (2iz)^{-k} + O\left(|z|^{-n-1}\right) \right] , \quad (42)
\]
where \( n \) is the order of the Hankel function and
\[
c_{n,k} = \frac{(4n^2-1)(4n^2-3^2)\cdots(4n^2-(2k-1)^2)}{2^{2k}k!}
\]
and
\[
c_{n,0} = 1
\]
(Lebedev, 1965, p 122). We can calculate \( H_0^{(2)} \) from this by noting that it is the complex conjugate of \( H_0^{(1)} \).

References


