Detection and imaging of moving objects with synthetic aperture radar

Part 2: Joint time–frequency analysis by Wigner–Ville distribution

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Abstract: The aim of the work is to show how time–frequency representation by Wigner–Ville distribution of the echoes received by a synthetic aperture radar provides a useful tool for detection of moving objects and the estimation of the instantaneous phase shift induced by relative radar–object motion. The phase history is then used to compensate the received signal and to form a synthetic aperture with respect to the moving object, necessary to produce a high resolution image.

1 Introduction

The main problems related to the detection and imaging of moving targets with synthetic aperture radars (SAR) are analysed in the first part of the paper. This second part is entirely focused on the use of the Wigner–Ville distribution (WVD) in the SAR signal processing for moving target detection and imaging.

The possibility of focusing a moving target observed by radar requires knowledge of the phase modulation induced on the received echo by the radar–target relative motion. The time behaviour and/or the spectrum of the target echoes alone are not sufficient to provide this information. The knowledge of the phase history of the echoes received during the whole observation interval allows us to form the synthetic aperture with respect to the moving object, necessary to produce a high cross-range resolution image.

A method for extracting the instantaneous phase can be based on analysis of the time–frequency (TF) distribution of the received signal. Several distributions are available and an excellent review is given in a recent paper by Cohen [1]. The Wigner–Ville distribution has been chosen in this work because it presents some important features concerning detection and estimation issues, as already pointed out [2–5]. There are simpler methods for analysing signals in the TF domain, such as the short-time Fourier transform (STFT) [1], but they do not exhibit the same resolution capabilities in the TF domain as does the WVD. In particular, since the STFT is based on a Fourier transform (FT) applied to a time windowed version of the signal, with the window central instant varying with time, the frequency resolution is inversely proportional to the window duration. The narrower the window, the better is the time resolution, but the worse is the frequency resolution and vice versa. Conversely, the WVD does not suffer from this shortcoming. On the other hand, the WVD poses other problems since it is not a linear transformation. This causes the appearance of undesired cross-products when more than one signal is present.

With respect to other TF distributions, such as Rihaczek’s (e.g. Reference 1), the WVD provides a higher concentration of the signal energy in the TF plane, around the curve of the signal instantaneous frequency (IF). This allows a better estimation of the IF in the presence of noise and this information is fundamental for the synthesis of the long aperture with respect to the moving object.

Mapping of the received signal in the TF plane provides a tool for the synthesis of the optimal receiver filter without a priori knowledge of the useful signal, provided that the signal-to-noise ratio be sufficiently large. The TF representation provides a unique tool for exploiting one of the most relevant differences between useful signals and disturbances in the imaging of small moving objects, namely the instantaneous frequency and the bandwidth. In fact, it can be shown that, while the bandwidth occupied by a target echo during the observation interval necessary to form the synthetic aperture mainly depends on radar–object motion, the instantaneous bandwidth is proportional to object size. Therefore the echo corresponding to a small target can occupy a large band during the overall observation time, but its instantaneous bandwidth is considerably narrower (i.e. the echo backscattered by a point-like target has a zero instantaneous bandwidth but it may exhibit a large overall bandwidth). Conversely, the echo from the background and the receiver noise have a large instantaneous bandwidth. Therefore, even if the useful signal and the disturbance may have a large total band, the possibility of tracking the instantaneous bandwidth, made available by the TF representation allows a discrimination of the useful signal from the disturbance not possible by conventional processing.

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Another important and unique advantage related to use of the WVD is that it allows the recovery of the echo phase history even in the case of undersampling, as shown in Reference 6. This is particularly important in SAR applications since it allows us to work with pulse repetition frequencies (PRF) lower than the limits imposed by the signal bandwidth occupied during the observation interval. Owing to the target motion, this bandwidth may be considerably larger than the bandwidth occupied by the background echo. According to conventional processing, we should then use a correspondingly higher PRF. Conversely, if the useful signal has a large total bandwidth, but a narrow instantaneous bandwidth, the TF representation prevents superposition of spectrum replicas created by undersampling because, even if the replicas occupy the same bandwidth, they occur at different times. This property allows us to recover the desired information even from undersampled signals. Since the PRF value imposes a limit on the size of the monitorable area, due to time, and then range, ambiguities, the possibility of using a low PRF prevents the reduction of the region to be imaged, as well as the increase of the data rate.

The WVD is first recalled in Section 2. In particular, its main properties in the presence of noise are derived. Section 3 is then concerned with recasting the optimal detection and parameter estimation in presence of noise in the TF domain, based on the WVD. Use of the WVD for the detection of chirp signals embedded in white noise has already been considered [2–5]. The approach is here extended to different signal modulations and to the case of correlated noise (the echo from the background). On the basis of the matched filter theory, transferred into the TF domain, it is shown that, in the presence of white noise, the received signal can be mapped directly onto the TF domain. In the presence of a correlated disturbance (the echo from the background), a cancellation must be performed before evaluating the WVD. A possible algorithm for filtering the received signal in the TF plane is also proposed. Finally, the application to the SAR case is considered in Section 4. Simulation results are shown for evaluating the capabilities of the proposed approach.

2 Time-frequency analysis by means of the Wigner–Ville distribution

2.1 Definition and main properties of the WVD

In a recent paper [1], a thorough overview of TF distributions is provided. An extensive analysis of the Wigner–Ville distribution is also given in References 7–9. The WVD of a signal is defined as:

\[ W(t, f) = \int_{-\infty}^{+\infty} W_{\phi}(t, f) df \]

where \( s(t) \) represents the analytic signal.

Two properties, particularly important in detection and estimation problems are: (a) conservation of the scalar or inner products; (b) estimation of the instantaneous frequency. These properties are now briefly recalled.

(a) Conservation of the scalar products: Given two signals \( x_i(t), i = 1, 2 \), the square modulus of their scalar product is equal to the scalar product between their distributions:

\[ \int_{-\infty}^{+\infty} x_1(t) \overline{x_2(t)} dt = \int_{-\infty}^{+\infty} W_{x_1}(t, f) \overline{W_{x_2}}(t, f) df \]

having indicated with \( W_{\phi}(t, \omega) \) the WVD of the signals \( x_i(t), i = 1, 2 \).

This property will provide the basis for the formulation of the detection scheme in the TF domain.

(b) Estimation of the instantaneous frequency: If we express the signal in terms of its envelope and phase:

\[ x(t) = a(t) \exp \{ j\omega(t) \} \]

it can be shown that the local or mean conditional frequency of the WVD distribution, defined as [1]

\[ \langle f \rangle_\lambda = \left\{ \int_{-\infty}^{+\infty} W(t, f) df \right\} \left\{ \int_{-\infty}^{+\infty} W(t, f) df \right\}^{-1} \]

is equal to the signal IF:

\[ \langle f \rangle_\lambda = \frac{1}{2\pi} \frac{\partial \omega(t)}{\partial t} \]

The estimate of the mean conditional frequency of the WVD then provides the information about the signal IF. This is exactly the information we need for rephasing the received signal in order to produce a focused image.

2.2 WVD of signal plus noise

A problem which arises when using the WVD is that, since it is not a linear transformation, the distribution of the sum of more than one signal is not equal to the sum of the distributions of each signal, but contains all the cross-products.

In general, if a signal \( s_i(t) \) is given by the sum of \( N \) contributions

\[ s_i(t) = \sum_{i=1}^{N} s_i(t) \]

its WVD is

\[ W_{\phi}(t, f) = \sum_{i=1}^{N} W_{\phi}(t, f) + 2 \sum_{i=1}^{N} \sum_{j=1}^{N} \text{Re} \{ W_{\phi}(t, f) \} \]

where \( W_{\phi}(t, f) \) is the cross-WVD of \( s_i(t) \) and \( s_j(t) \):

\[ W_{\phi}(t, f) = \int_{-\infty}^{+\infty} s_i(t) \overline{s_j(t)} \exp \left( -j\frac{\pi}{2} \right) \exp \left( -j2\pi ft \right) dt \]

In detection problems, it is important to analyse the main statistical properties of the WVD of deterministic signals superimposed on stochastic processes (the disturbance). Therefore it is useful to review the main properties of the WVD of a random process.

In the case of a stationary random process \( c(t) \), the expected value of the WVD is constant over the time axis and is equal to the power spectral density of the process over the frequency axis. In fact:

\[ E\{ W_{\phi}(t, f) \} = \int_{-\infty}^{+\infty} E\{ c(t + \frac{\tau}{2}) \overline{c(t - \frac{\tau}{2})} \exp \left( -j2\pi ft \right) dt \]

\[ = \int_{-\infty}^{+\infty} R_c(\tau) \exp \left( -j2\pi ft \right) dt = S_c(f) \]

where \( R_c(\tau) \) and \( S_c(f) \) are the correlation and the power spectral density of the random process, respectively. In the case of white Gaussian noise, the WVD is constant.

In the case of a deterministic signal superimposed on a zero mean additive noise, the cross-product term has a zero mean value. The corresponding average WVD is then equal to the sum of the WVD of the signal plus the
average WVD of the noise:
\[
E[W_{s+n}(t,f)] = W(t,f) + E[W_d(t,f)]
\]
\[
= W(t,f) + S_d(f)
\]  
(9)

Fig. 1  WVD of a linear frequency modulation signal

\(a\) Contour levels
\(b\) Perspective view

where \(S_d(f)\) is the noise power spectral density. This does not mean that there are no cross-products between signal and noise of course, it only means that these terms have a zero mean value.

2.3 Estimation of the instantaneous frequency in the presence of noise or more than one signal

Given two deterministic signals \(s_1(t)\) and \(s_2(t)\) with constant envelope:

\[
s_i(t) = A_i \exp \{j\phi_i(t)\} \quad i = 1, 2
\]  
(10)

the IF of their sum, estimated by using eqn. 4, is (see Appendix):

\[
\begin{align*}
A_1^2 f_1(t) + A_2^2 f_2(t) + A_1 A_2 f_{12}(t) \\
A_1^2 + A_2^2 + 2A_1 A_2 \cos (\phi_1(t) - \phi_2(t))
\end{align*}
\]

\[
f_{12}(t) = \frac{f_1(t) + f_2(t)}{2}
\]  
(11)

In general, this expression is not easily interpretable. Two extreme cases can be considered:

- (a) \(A_1 = A_2 = A\):

\[
f_{12}(t) = f(t)
\]  
(12)

- (b) \(A_1 > A_2\):

\[
f_{12}(t) = f_1(t) \quad \text{(zero-order approximation)}
\]

\[
f_{12}(t) = f_1(t) + (A_2/A_1) [ f_2(t) - f_1(t) ] \cos [\phi_1(t) - \phi_2(t)]
\]  
(13)

If the two signals have the same amplitude, the estimated frequency lies exactly in the middle between the two instantaneous frequencies, otherwise it lies between them, in a position depending on the relative amplitudes and on their phase modulation. If the signal phases are independent random variables evenly distributed in \([0, 2\pi]\), the average value of \(f_{12}(t)\) is approximately equal to the centre of gravity of the various IF components, weighted by the corresponding power:

\[
E[f_{12}(t)] = \frac{A_1^2 f_1(t) + A_2^2 f_2(t)}{A_1^2 + A_2^2}
\]

(14)

The presence of noise inevitably affects the estimation of the IF. In fact, in such a case, we have:

\[
\int_{-\infty}^{+\infty} f_{W_s+n}(t,f) \, df = \int_{-\infty}^{+\infty} f_{W_s}(t,f) \, df + \int_{-\infty}^{+\infty} f_{S_d}(f) \, df
\]

\[
= \int_{-\infty}^{+\infty} f_{W_s}(t,f) \, df + \frac{a^2(t) \phi(t)}{2\pi}
\]  
(15)

since \(S_d(f)\) is an even function. Furthermore

\[
\int_{-\infty}^{+\infty} E[W_s+n(t,f)] \, df
\]

\[
= \int_{-\infty}^{+\infty} W_s(t,f) \, df + \int_{-\infty}^{+\infty} S_d(f) \, df
\]

\[
= a^2(t) + P_n
\]  
(16)

where \(P_n\) represents the noise power. Consequently, as a first approximation, we have:

\[
E\{\langle f \rangle_s\} = \frac{1}{2\pi} \frac{a^2(t) \phi(t)}{a^2(t) + P_n}
\]

(17)

For a signal with constant amplitude \(A\), the expected value of the instantaneous frequency is then:

\[
E\{\langle f \rangle_s\} = \frac{A^2 \phi(t)}{2\pi A^2 + P_n}
\]

\[
= \frac{\phi(t)}{2\pi + P_n/A^2}
\]

(18)

having indicated by SNR the signal-to-noise ratio. From this expression it turns out that, in presence of noise, the average value is always smaller than the true value.

2.4 Examples

Some examples of distribution can be helpful to understand the representation by WVD. An important example in SAR applications is a signal with a constant envelope and a phase modulation law composed by the superposition of a polynomial term (slow time fluctuations) plus sinusoidal contributions (fast fluctuations):

\[
s(t) = A \exp \left\{ -j[\phi_0 + 2\pi f_d t + \pi \mu t^2 \cos \omega_0 t ] \right\} \quad |t| < T/2
\]

(19)

It is useful to analyze separately the effect of the different phase contributions on the WVD.
(a) Linear frequency modulation ($f_d \neq 0, \mu_d \neq 0, 
\eta_d = 0, k = 0$): In this case, the WVD is known in closed form:

$$W(t,f) = 2A^2(T - 2|t|) \text{sinc} \left(2\pi(T - 2|t|)\right) \times (f - f_d - \mu_d t) \hspace{1cm} |t| < T/2$$

(20)

![Figure 2](image1.png)  
**Figure 2**  
Instantaneous frequency estimated from the WVD of Fig. 1

![Figure 3](image2.png)  
**Figure 3**  
WVD of a signal with cubic and sinusoidal phase modulation
a Contour levels  
b Perspective view

Fig. 1 shows the WVD of this signal, represented in terms of contour level curves $a$ and in perspective $b$. In Fig. 2, the IF estimated by the previous distribution, using eqn. 4, is reported. The oscillations of the estimated value around the true value and the consequent error present at the borders of the time window are due to the fact that, at the borders, owing to the finite time duration of the signals, there are not enough samples to yield an accurate estimation.

(b) Parabolic plus harmonic modulation ($f_d \neq 0, \mu_d \neq 
0, \eta_d \neq 0, k \neq 0$): The WVD of this kind of signal is sketched in Fig. 3, and the relative instantaneous frequency in Fig. 4. With respect to the previous case, the WVD is more spread out over the TF plane, but the estimated frequency still follows the right behaviour.

3 Detection and parameter estimation of signals embedded in noise based on the WVD

3.1 Rationale for using the WVD in detection and estimation problems

The received signal can be modelled as the sum of a useful signal (the echo from the moving object), whose phase modulation is unknown, plus a correlated random process or clutter (the echo from the fixed background), plus noise. The echoes from the fixed scene represent, in our case, a disturbance. Our aim is to detect the presence of moving objects and to estimate their motion parameters. The detection and parameter estimation performance depend on the signal-to-clutter ratio (SCR). If this ratio is too small, we have to process first the received signal, in order to improve it as much as possible. This operation can be carried out by matched filtering. However, the matched filter can be defined only if the shape of the useful signal is known, and this is not the case. It turns out that the two operations, detection and parameter estimation, cannot be separated: estimation of the useful signal parameters can be carried out only after having detected the presence of a useful signal; a reliable detection, on the other hand, requires the knowledge of the signal parameters, in order to carry out a proper matched filtering, before the detection itself. It is then necessary to carry on these two kinds of operations contemporaneously.

The time–frequency analysis of the received signal, in particular the computation of its Wigner–Ville distribution, provides a powerful tool for achieving the aforementioned requirements and extracting the desired information, namely the energy and the phase history. These two kinds of information are what we need for our purposes: the detection of the presence of a moving target is made by comparing the energy with a suitable threshold; the instantaneous phase is used to phase-compensate the received signal, for a correct coherent integration, necessary for imaging purposes.
As regards the effect of disturbances in the received signal, it is useful to recall that matched filtering can always be performed by cascading two kinds of operations: a clutter cancellation followed by a coherent integration (see the first part of the paper). The first operation does not require knowledge of the useful signal shape, whereas the second operation does. In particular, a correct coherent integration requires knowledge of the signal phase history. The time–frequency analysis aims to facilitate extraction of the signal phase history. Therefore, the sequence of operations to be performed on the received signals is the following: a clutter cancellation is performed first and the output of the cancellation filter is analysed in the TF domain. If the clutter has been reduced to a power level sufficiently smaller than the useful signal, the parameters estimated by the WVD are correct, within an error depending on the achieved SCR. These parameters provide the information necessary to set up a correct coherent integration.

This rather intuitive reasoning for using the WVD in detection and estimation problems is now formalised into the framework of the matched filter theory.

### 3.2 Matched filtering based on the WVD

The received signal can be modelled as the sum of a useful signal \( a(t) \) (supposed known in a parametric form, but this restriction will be removed later on), plus a Gaussian process with zero mean value and autocorrelation matrix \( R_x \) (see Part 1). The optimal detection scheme consists of taking the inner product between the vector \( r \), containing samples of the received signal, and the optimal weighting vector \( w(\theta) \), given by:

\[
w(\theta) = R_x^{-\frac{1}{2}} s(\theta)
\]

where \( s(\theta) \) is the vector formed by samples of the useful signal, supposed known in parametric form, and the vector \( \theta \) contains the unknown signal parameters. The maximum modulus of all the scalar products

\[
x(\theta) = r^T \cdot w(\theta)
\]

as a function of the parameters is then compared with a threshold to determine the presence of a useful signal.

By using the property of conservation of the inner products (eqn. 2) for Wigner–Ville distributions, the square modulus of \( y \) can be evaluated alternatively as the inner product between the WVD of \( r \) and the WVD of \( w \). This allows us to recast the matched filter theory in the TF domain.

The case of an uncorrelated disturbance is now examined, and then the theory will be extended to correlated disturbances.

#### 3.2.1 White noise case

In the case of a white noise, the correlation matrix \( R_x \) is diagonal, therefore the weighting vector is directly proportional to the useful signal. Detection is carried out by thresholding the modulus of the scalar product between the received vector and the conjugated value of the useful signal. According to eqn. 2, the square modulus of that scalar product is

\[
|y(\theta)|^2 = \left| \int_{-\infty}^{+\infty} r(t)s^*(t; \theta) dt \right|^2
= \int_{-\infty}^{+\infty} \left| W_r(t; f)W_x(t; f; \theta) df \right|^2
\]

According to the TF formulation, we evaluate the WVD of the received signal \( W_r(t; f) \) and the WVD of the useful signal \( W_x(t; f) \), take their inner product and compare the result with a threshold. Since the energy distribution of the signals of interest is concentrated around the curve of the signal IF, the double integral can be reduced to a line integral.

In the case of a chirp signal, the energy is concentrated along a straight line in the TF plane (see Fig. 1a). The vector \( \theta \) in this case is composed by two parameters: the mean Doppler frequency \( f_p \) and the Doppler rate \( \mu_d \). The parameters of the line of maximum energy, constant term and slope, coincide exactly with the chirp parameters. These parameters can be estimated by analysing the TF distribution of the received signal. As suggested in Reference 3, the detection and estimation would consist, in this case, in the evaluation of all the line integrals over the TF plane, (by varying the line parameters \( f_p \) and \( \mu_d \)), the search for a maximum, and the comparison with a threshold. The parameters of the line which gives the maximum energy are exactly the parameters of the target IF. The WVD then offers a means for simultaneously detecting and estimating the modulation parameters of the chirp.

When the SNR is not too small, it is possible to simplify the aforementioned approach as follows: (i) compute the WVD of the received signal (ii) estimate the signal IF \( f(t) \) by computing the centre of gravity of the WVD, for each time instant, by eqn. 4 (iii) evaluate the instantaneous phase, apart from an unimportant constant term, by simply integrating the estimated frequency:

\[
\phi(t) = 2\pi \int_{t_0}^{t} f(\tau) d\tau + \phi(t_0)
\]

Finally, (iv) generate a reference signal:

\[
s(t) = \exp \{-j\phi(t)\}
\]

and phase-compensate the received signal by mixing it with this reference signal.

The great advantage of this approach is that it does not require any assumption about the kind of phase modulation. In particular, we can analyse signals whose frequency modulations do not necessarily follow a straight line.

We would point out that the simplification in the estimation procedure is achievable only if the signal-to-noise ratio is sufficiently large, otherwise the frequency estimation could be seriously degraded. However, a smoothing of the WVD samples can be used for improving the estimation accuracy in the presence of noise, as will be shown in Section 3.3.

#### 3.2.2 Correlated noise case

The strategy followed in Section 3.2.1 can be applied to this case by simply rearranging the terms in eqn. 22. In fact, we can write eqn. 22 as:

\[
x(\theta) = r^T (R_x^{-\frac{1}{2}} s^*(\theta)) = (R_x^{-T})^T x(\theta) = x^T s^*(\theta)
\]

where the superscript \( -T \) indicates inverse and transpose. Therefore, the square modulus of \( y(\theta) \) can be evaluated as the scalar product between the WVD of \( x \) and the WVD of \( s \). At this point, the procedure followed in the white noise case can be applied to this case as well, provided that the vector \( r \) is substituted by the vector \( x \). Since the multiplication of the vector \( r \) by the inverse of the clutter correlation matrix corresponds to a clutter cancellation (see the first part of the paper) this means that, as might be expected, the estimation procedure must be applied not directly on the received data, as in the white noise case, but on the data at the output of a clutter cancellation filter.
3.3 Instantaneous frequency estimation in presence of noise

According to classical estimation theory, based on the maximum likelihood principle, the signal parameters can be estimated as that set of parameters which maximise the likelihood ratio or, equivalently for Gaussian-distributed processes, eqn. 23. The bottleneck of this technique is that the likelihood ratio is a multimode nonlinear function of the parameters to be estimated. Therefore, as in any nonlinear optimisation problem, the solution is hard to obtain if the number of parameters is large, and the search can lead to a local maximum.

The TF representation makes the estimation problem easier. In fact, since the WVD provides a distribution whose energy is concentrated along the curve of the IF, it makes the extraction of the frequency modulation parameters easier. Of course, the accuracy of the estimation depends on the SNR.

In the presence of noise, some smoothing or filtering has to be applied to reduce the effect of the noise on the estimation accuracy. Different kinds of filtering can be envisaged, in the time domain or in the TF domain, that is directly on the received time samples or on their TF distribution. Furthermore, different strategies have to be followed depending on the noise correlation properties.

As shown in the previous Section, the parameter estimation is applied on the received time samples directly, in the presence of white noise, while a clutter cancellation filter has to be used first in the case of a correlated disturbance. However, the filtering modifies the signal modulation behaviour and the consequent estimation. In the following, a strategy for improving the estimation accuracy in the presence of white noise is proposed. Then, the correlated noise case is analysed, and in particular the effect of the clutter filtering on the successive IF estimation.

3.3.1 White noise case: The WVD provides a powerful tool for the estimation of the signal IF in the presence of white noise since it allows us to exploit one of the most relevant differences between signal and noise, that is their IF behaviours. While the bandwidth occupied by the target echo during the overall observation interval may be large, the instantaneous bandwidth is usually much narrower than the total bandwidth. Conversely, a white noise exhibits both total and instantaneous large bandwidths. Therefore an energy concentration in the TF plane reveals the presence of a useful signal.

On the basis of this difference, an iterative procedure can be envisaged for improving the estimation accuracy, based on the following steps:

(i) the WVD of the received signal is evaluated
(ii) the instantaneous frequency \( f(t) \) and the instantaneous bandwidth \( B(t) \) are estimated by evaluating the mean and the standard deviation of the distribution along the frequency axis, for each time instant
(iii) the WVD samples are multiplied by a window in the TF plane equal to one within an interval centred around \( f(t) \) and having a width proportional to \( B(t) \), and zero outside
(iv) steps (ii) and (iii) are repeated until some convergence on the estimated IF is observed.

The rationale of the procedure is that, if the window is correctly positioned, some noise contributions are removed in step (iii), while the signal is retained, thus improving the SNR. Therefore, at the second iteration of the procedure, the estimation of \( f(t) \) and \( B(t) \) improves.

This allows the synthesis of a narrower window which, in turn, produces a further noise rejection. Therefore, as long as the windows are correctly positioned (which means that no useful signal contributions are discarded), the estimation accuracy improves at each step. Final accuracy is dictated, at least in principle, only by the noise falling within the signal instantaneous bandwidth, instead of by the noise falling within the band occupied during the whole observation interval. The method works properly if the initial estimate is accurate enough to position the window correctly.

3.3.2 Correlated noise case: In the presence of a correlated noise, the received signal must be filtered to cancel the disturbance, before mapping the signal onto the TF domain. Of course, the filtering modifies the signal behaviour. In particular, the filtering attenuates all the signal components falling within the clutter band. Therefore, it is possible to recover the desired information only if the signal spectrum is not entirely overlapped with the clutter spectrum. Once having cancelled the clutter, the same procedure outlined for the white noise case can be applied to this case.

4 Application to detection and focusing of moving objects with SAR

4.1 Echo model

The signal received by a SAR is given by the sum of the echoes from the ground, the echoes from a possible moving object and receiver thermal noise. The echo from the ground can be modelled as a correlated random process, whose power spectral density is proportional to the antenna power radiation pattern (see Part I). The echo from a moving object is characterised by an unknown modulation, induced by the relative motion between the radar and the object. In SAR imaging, we are interested in the phase modulation induced by the relative radar–target motion. If the transmitted signal is

\[
p(t) = a(t) \exp(j2\pi f_c t)
\]

where \( a(t) \) is the analytic signal and \( f_c \) is the carrier frequency, the echo received by a pointlike scatterer at a distance \( r(t) \) from the radar is proportional to

\[
p(\frac{t - 2r(t)}{c}) = a(\frac{t - 2r(t)}{c}) \exp(j2\pi f_c t) \exp(j\frac{4\pi}{\lambda} r(t))
\]

During the observation interval, the amplitude of each backscattering coefficient can be assumed constant since the aspect angle does not vary by an amount such as to justify a change in the reflectivity characteristics (this assumption underlies all SAR signal processing). The only modulation of interest then is phase modulation.

In the formation of the synthetic aperture, we are interested in the last term of this expression. From Kirk's work on motion compensation for SAR [10], it turns out that the motion fluctuations which need to be compensated have periods comparable with the observation time (the analysis developed in Reference 10 mentions an SAR observing a fixed scene, but the same considerations can be applied to the imaging of moving objects, provided that the relative radar–target motion is taken into account). The conventional techniques for autofocus extended scenes compensate only linear and quadratic phase shifts. This means that only slow fluctuations (with
respect to the integration time) are compensated. Conversely, the TF approach allows estimation and compensation of any kind of phase history [11].

The distance \( r(t) \) can be modelled as the sum of slow and rapid fluctuations, with respect to the observation interval. Slow variations can be expressed as a low-order polynomial, whereas fast variations follow a sinusoidal behaviour, whose period is shorter than the time observation interval.

The change in the distance causes a phase modulation, which must be compensated to obtain a correct image. The variation of distance can cause migration of the received echo over different range cells, depending on the ratio between the amount of variation and the range resolution. The range migration problem can be faced according to the double-resolution strategy outlined in the Introduction to Part 1 of this paper. According to that strategy, all the phase estimations are carried out on data whose range resolution is such as to consider the migration negligible. Examples of WVDs of clutter signal at the input and at the output of the clutter canceller are reported in Figs. 6 and 7, respectively (the clutter has been generated as a random process with a given power spectral density). The example is relative to the case of a pulse repetition frequency (PRF) equal to 16 times the clutter bandwidth. The number of samples is 64 and the signal-to-clutter ratio (SCR) at the input is 0 dB. The signal has a linear frequency modulation with a bandwidth, due to the Doppler rate, equal to PRF/8. The mean Doppler frequency has been chosen equal to PRF/8 (in the analysis of these distributions, it is worth recalling that the discrete WVD is periodic of PRF/2 [8]).

**Fig. 6** WVD of clutter plus signal at the input of the clutter cancellation filter

\( \text{SCR} = 0 \text{ dB} \)

**Fig. 5** Block diagram of the detection and focusing scheme
From these Figures, it is evident that the filtering has improved the signal-to-clutter ratio. Since the processing procedure is based on the theory of optimal filtering, the achievable improvement in the signal-to-clutter ratio (SCR) can be deduced from the optimal performance shown in Part I.

The filter has also modified the shape of the signal distribution. In particular, the signal spectrum falling within the clutter spectrum is strongly attenuated. The resulting distribution exhibits a mean Doppler frequency higher than the true value and a Doppler slope very close to that of the undistorted signal. Since the focusing depends only on the Doppler slope, but is not affected by the mean Doppler frequency (which affects only the relative position of the target image with respect to the background) the successive estimation of the modulation parameters still produces a focused image.

An example of an image relating to a moving point-like target superimposed on a fixed extended scene is shown in Fig. 8. The image obtained after clutter cancellation is shown in Fig. 9. The target signal has barely come out, but it is still smeared in crossrange. By comparing the modulus of each output sample with a threshold obtained by averaging the moduli of the contiguous cells, as in the cell-averaging CFAR (constant false alarm rate) thresholding scheme, we can detect the presence of a moving target on the corresponding range cell. That range cell is analysed by the proposed approach and the result is shown in Fig. 10 (relative to the only range cell of interest). The sharpening of the target image is quite evident.

5 Conclusions

In this paper a scheme has been proposed for detecting and focusing moving targets with synthetic aperture radars based on a joint TF analysis of the received signals. The TF formulation has provided us with a powerful tool for getting the desired information about the useful signal instantaneous frequency, which is what we need for producing a focused image of the target. The method works for point-like targets (targets whose dimensions are smaller than the radar resolution) or for extended targets characterised by a dominant scatterer or by independent scatterers having similar reflectivity characteristics. The method provides a compensation of the relative radar–target translational motion, whatever it would be. Once having compensated the translational motion, the remaining rotational motion can be compensated by, for example, the method proposed in Reference 12.

The range migration problem has been overcome using the double-resolution strategy outlined in the first part of the paper. The WVD is used to estimate the instantaneous frequency, and then the instantaneous phase, on the low-range resolution data. The estimation is then used to recover the law of variation of the radar–target distance, to be used in the compensation of the range migration on the full range resolution data.

Further analyses are necessary in the imaging of extended targets when more dominant scatterers occupy the same resolution cell. In such a case, in fact, the bilinearity of the WVD creates undesired crossproduct terms which can seriously impair the estimation of the instantaneous frequency. A reduction of these undesired terms could be achieved by resorting to other time–frequency representations, such as the Choi–Williams distribution [11], for example, or by applying a Radon or a Hough transform on the WVD. Since the received signal is intrinsically two-dimensional (range and crossrange), it is
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7 References


8 Appendix: Estimation of the instantaneous frequency of the sum of several signals

Given a signal s(t) equal to the sum of two signals s1(t) and s2(t), expressed in terms of their envelopes and phase:

\[ s_i(t) = a_i(t) \exp\{j\phi_i(t)\} \quad i = 1, 2 \]  

(29)

its WVD is (see eqn. 7):

\[ W_{1 + 2}(t,f) = W_1(t,f) + W_2(t,f) + 2 \Re\{W_{12}(t,f)\} \]  

(30)

where we have indicated by \( W_{1 + 2}(t,f) \) the WVD of the sum, by \( W_1(t,f) \) the WVD of the signal \( s_1(t) \), and by \( W_{12}(t,f) \) the cross-WVD:

\[ W_{12}(t,f) = \int_{-\infty}^{+\infty} s_1(t + \frac{\tau}{2}) \overline{s_2(t - \frac{\tau}{2})} \exp\{-j2\pi ft\} \, d\tau \]  

(31)

The instantaneous frequency of the sum signal is evaluated by using eqn. 4. In particular, the numerator of eqn. 4 in this case is equal to:

\[ \int_{-\infty}^{+\infty} f W_{1 + 2}(t,f) \, df \]

\[ = \int_{-\infty}^{+\infty} f W_1(t,f) \, df + \int_{-\infty}^{+\infty} f W_2(t,f) \, df \]

\[ + 2 \Re\left\{ \int_{-\infty}^{+\infty} f W_{12}(t,f) \, df \right\} \]

\[ = a_1^2(t)f_1(t) + a_2^2(t)f_2(t) \]

\[ + 2 \Re\left\{ \int_{-\infty}^{+\infty} f W_{12}(t,f) \, df \right\} \]  

(32)

where \( f_1(t) \) and \( f_2(t) \) are the instantaneous frequencies of the two separated signals \( s_1(t) \) and \( s_2(t) \). The contribution due to the crossproduct terms is:

\[ \int_{-\infty}^{+\infty} f W_{12}(t,f) \, df = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(s_1(t + \tau/2)s_2^*(t - \tau/2) \times \exp\{-j2\pi ft\} \, dt \]

By exploiting the derivation property of the Fourier transform, we can write this expression in the following form:

\[ \int_{-\infty}^{+\infty} f W_{12}(t,f) \, df = \frac{1}{j2\pi} \left[ a_1^2(t)a_2^2(t) - a_1(t)a_2(t)\phi_1(t) + a_1(t)a_2(t)\phi_2(t) \right] \times \exp\{j[\phi_1(t) - \phi_2(t)]\} \]  

(33)

where the prime indicates a derivative.

In the case of constant envelopes \( a_1(t) = A_1 \) and \( a_2(t) = A_2 \), taking the real part, according to eqn. 32, we have:

\[ \int_{-\infty}^{+\infty} f W_{12}(t,f) \, df \]

\[ \quad = A_1A_2[|f_1(t) + f_2(t)|/2] \cos[\phi_1(t) - \phi_2(t)] \]  

(34)

The numerator of the estimated instantaneous frequency is then:

\[ \int_{-\infty}^{+\infty} f W_{1 + 2}(t,f) \, df \]

\[ = A_1^2f_1(t) + A_2^2f_2(t) \]

\[ + A_1A_2[f_1(t) + f_2(t)] \cos[\phi_1(t) - \phi_2(t)] \]

By proceeding in a similar way for the denominator, we obtain:

\[ \int_{-\infty}^{+\infty} W_{1 + 2}(t,f) \, df \]

\[ = A_1^2 + A_2^2 + 2A_1A_2 \cos[\phi_1(t) - \phi_2(t)] \]

The instantaneous frequency is then

\[ A_1^2f_1(t) + A_2^2f_2(t) + A_1A_2[f_1(t) + f_2(t)] \times \cos[\phi_1(t) - \phi_2(t)] \]  

(35)